# Introduction

## Lecture 1, 21-02-2024

The origins of the theory stem from Felix Klein's "Erlangen programm"

Idea: geometry of a space is determined by its group of symmetries.

Lie groups are named after Sophus Lie.

Lie's idea: develop a theory of symmetries of differential equations.
~ Galois theory for algebraic equations.

Lie groups are an indispensable tool in several branches of mathematics and theoretical physics.

Roughly speaking: a Lie group is a differentiable manifold that is also a group and such that the group operations are compatible.

with the manifold structure.
(will discuss precise definition later in the course)

To any Lie group we can associate an algebraic object, the Lie algebra of the group.
Geometrically, it corresponds to the tangent space of the Lie group at the identity.

In the famous problem list publicized by David Hilbert in 1900, the fifth problem asked if there was any difference in assuming that the topological space underlying the group was only a topological manifold.

The answer is that there is no difference, and it came with the work of Andrew Gleason, Deane Montgomery and Leo Zippin in the early 1950s.

The interpretation of the original question of Hilbert is debated.

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A stronger interpretation with respect to the one mentioned above leads to the so-called:

Hilbert-Smith conjecture. (1941)

If a locally compact group G acts faithfully and continuously on a topological manifold M, then G is a Lie group.

The statement is true in dimension $\leq 2$ (classical) and it was proved in dimension 3 by John Pardon in 2011. In general it is completely open.

[Montgomery-Zippin 1955]

Structure of the course.

The course will be divided in three parts

- Topological groups, homogeneous spaces and invariant measures
- Lie groups and their Lie algebras; correspondence between subalgebras and subgroups.
- Structure theory.

In the first part we shall see how far one can go if $G$ is only assumed to be a topological space and the operations are continuous. (The answer will be that we can go surprisingly far)

In the last part we will try to decompose Lie groups and Lie algebras into fundamental blocks that are easier to understand.
$\leadsto$ Will generalize some classical decomposition results for matrices in linear algebra.

Note: the resolution of Hilbert's fifth problem is particularly striking, because:

[Milnor '56]: there are topological manifolds that admit multiple non-diffeomorphic smooth structures

[Freedman '82]: there exists a top. manifold that does not admit any smooth structure.

# Topological groups (Chapter 2).

## 2.1 Definitions and examples.

**Def 2.1**
A topological group is a group $G$ whose underlying set is endowed with a topology s.t the multiplication:

$$m : G \times G \longrightarrow G.$$
$$(g, h) \longmapsto gh.$$

and the inverse.

$$i : G \longrightarrow G$$
$$g \longmapsto g^{-1}$$

are continuous maps,

**Remark** : above. $G \times G$ is endowed with the product topology

## 2.2. Elementary consequences

i). Since $i : G \longrightarrow G$ is continuous. and $i \circ i = id_G$, $i$ is a homeomorphism.

(i) For every $g \in G$ the left translation $L_g : G \longrightarrow G$, $x \longmapsto gx$

and the right translation $R_g : G \longrightarrow G$, $x \longmapsto xg$, are continuous. Moreover it is obvious that

$$L_g \circ L_{g^{-1}} = L_{g^{-1}} \circ L_g = id_G$$

and

$$R_g \circ R_{g^{-1}} = R_{g^{-1}} \circ R_g = id_G$$

Hence $L_g$ and $R_g$ are homeomorphisms.

(ii) Let $\varphi : G_1 \longrightarrow G_2$ be a homomorphism of topological groups.;

(*) $\varphi(gh) = \varphi(g) \cdot \varphi(h) \quad \forall g, h \in G_1$

The condition (*) can be expressed as commutativity of the diagram

$$\begin{tikzcd} G_1 \arrow[r, "\varphi"] & G_2 \arrow[d, dash, "L_{\varphi(g)}"] \\ {} \arrow[u, "L_{g^{-1}}"] & {} \end{tikzcd}$$

$$G_1 \xrightarrow{\varphi} G_2$$

for all $g \in G_1$.

$\boxed{\text{Consequence}}$: if $\varphi$ is continuous at $e \in G_1$ then it is continuous on $G_1$

iv) A subgroup $H \leq G$ of a top. group $G$ is a topological group when equipped with the induced topology

v) Given a family of top. groups $G_\alpha$, $\alpha \in A$ the cartesian product $\prod_{\alpha \in A} G_\alpha$ with the product topology is a top. group.

vi) If $H \triangleleft G$ is a normal subgroup of a top. group. then $G/H$ (quotient) endowed with the quotient topology is a topological group.

$\boxed{\text{Example 2.3.}}$

Any group $G$ with the discrete topology.

**Example 2.4**

$\mathbb{R}^n$ with the Euclidean topology and addition $+$.

**Example 2.5**

The multiplicative groups $\mathbb{R}^*$ and $\mathbb{C}^*$ of the fields $\mathbb{R}$ and $\mathbb{C}$

**Example 2.6**

$M_{n,n}(\mathbb{R})$ space of $n \times n$ matrices with the Euclidean topology.

Note that the matrix product

$$(A,B) \longmapsto A \cdot B.$$

is continuous. Let:

$$GL(n,\mathbb{R}) := \{A \in M_{n,n}(\mathbb{R}) : \det A \neq 0\}$$

and note that it is an open subset of $M_{n,n}(\mathbb{R})$.

Then $GL(n,\mathbb{R})$ is a group with neutral element $Id$.

Use the formula $(A^{-1})_{ij} = \dfrac{\det M_{ij}}{\det A}$.

it is immediate to check that $GL(n,\mathbb{R})$ is a topological group.

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Consider a locally compact Hausdorff top. space $X$. We endow $\mathrm{Homeo}(X)$ (= group of homeomorphisms of $X$) with the compact - open topology.

Definition 2.7

Let $X, Y$ be topological spaces. The sets

$$S(C,U) := \{ f \in C(X,Y) : f(C) \subset U \}$$

where $C \subseteq X$ is compact and $U \subseteq Y$ is open form a sub-basis of the compact - open topology on $C(X,Y)$.

Definition 2.8

A sub-basis $\mathcal{S}$ of a topology $\tau \subseteq \mathcal{P}(X)$ on a set $X$ is a family of sets s.t. the family $\mathcal{B}$ obtained by taking all finite intersections of elements of $\mathcal{S}$ is basis.

**Exercise 2.9**

Homeo $(X)$ is <u>not</u> necessarily a top. group.

**Exercise 2.10**

Homeo $(X)$ <u>is</u> a top. group if $X$ is compact.

**Example 2.11**

If $X$ is locally compact Hausdorff and loc. connected. then Homeo $(X)$ <u>is</u> a top. group.

**Remark 2.12**

Topological manifolds are loc. compact, Hausdorff and loc. connected.

**Example 2.13**

Let $M$ be a smooth manifold $(C^\infty)$.

$$\mathrm{Diff}^r(M) := \{ f \in \mathrm{Homeo}(M) : f, f^{-1} \in C^r \}$$

is a subgroup of Homeo $(M)$ and hence it

is a top. group.
It is not closed in $\mathrm{Homeo}(M)$ wrt the compact open topology.

**Example 2.14**
Let $(X,d)$ be a proper metric space. ( = closed balls are compact ). Then its group of isometries

$$\mathrm{Iso}(X) := \{ f : X \to X \text{ bijection with } d(f(x), f(y)) = d(x,y) \ \forall x,y \in X \}$$

with the compact-open topology is a top. group.

**Remark 2.15**
For a metric space $(X,d)$ the compact open topology on $C(X,X)$ is the topology of uniform convergence on compact sets.

## Example 2.16

We consider some subgroups of $GL(n,\mathbb{R})$

(i).

$$A := \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{R}_{>0} \right\}.$$

is a closed subgroup of $GL(n,\mathbb{R})$, isomorphic to $(\mathbb{R}_{>0})^n$ as top. group.

(ii). The group of upper triangular matrices with 1s on the diagonal.

$$N := \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in GL(n,\mathbb{R}) \right\}$$

is a closed subgroup of $GL(n,\mathbb{R})$

Note. : $N$ is homeomorphic to $\mathbb{R}^{\frac{n(n-1)}{2}}$.
but it is not isomorphic to $\mathbb{R}^{\frac{n(n-1)}{2}}$.

Indeed if $n \geq 3$ then $N$ is not abelian.

(iii) The group. $K = O(n,\mathbb{R}) = \{ A \in GL(n,\mathbb{R}) : A \cdot {}^tA = \mathrm{Id} \}$ of orthogonal matrices.
is a closed subgroup of $GL(n,\mathbb{R})$

Note: the Gram-Schmidt orthogonalization procedure can be rephrased by saying that any matrix $g \in GL(n,\mathbb{R})$ can be written uniquely as

$$g = k \cdot a \cdot n.$$

with $k \in O(n,\mathbb{R})$, $a \in A$, $n \in N$.

It is possible to check that the map

$$K \times A \times N \longrightarrow GL(n,\mathbb{R})$$
$$(k, a, n) \longmapsto kan$$

is a homeomorphism.

In the last part of the course one of the goals will be to discuss generalizations of this kind of decomposition.

**Example 2.17**

Consider the symmetric bilinear form on $\mathbb{R}^n$.

$$B(x,y) := -\sum_{i=1}^{p} x_i y_i + \sum_{j=p+1}^{n} x_j y_j$$

$$= {}^t x \begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_q \end{pmatrix} y.$$

with $p+q=n$. Up to change of basis this is the unique symmetric non-degenerate bilinear form of signature $(p,q)$.

The group $O(p,q) := \{ g \in GL(n,\mathbb{R}) : g \text{ preserves } B \}$

$$= \left\{ g \in GL(n,\mathbb{R}) : {}^t g \begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_q \end{pmatrix} g = \begin{pmatrix} -\mathrm{Id}_p & 0 \\ 0 & \mathrm{Id}_q \end{pmatrix} \right\}$$

is a closed subgroup of $GL(n,\mathbb{R})$

**Note:** the Minkowski metric, which is key for special relativity corresponds to the choice $(p,q) = (1,3)$.

**Example 2.18**

In analogy with what done in the real case we define:

$$GL(n,\mathbb{C}) := \{ X \in M_{n,n}(\mathbb{C}) : \det X \neq 0 \}$$

$GL(n,\mathbb{C}) \subseteq M_{n,n}(\mathbb{C})$ is open and it is a top. group with the induced topology.

For $X \in M_{n,n}(\mathbb{C})$ we let $X^* := {}^t\overline{X}$

Then we can define (cf. with Ex 2.16).

$$A := \left\{ \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} : \lambda_i \in \mathbb{R}_{>0} \right\}$$

$$N := \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : * \in \mathbb{C} \right\}.$$

$$U(n) := \{ g \in GL(n,\mathbb{C}) : g g^* = \mathrm{Id} \}$$
$\leadsto$ unitary group.

Then there is a homeomorphism.

$$K \times A \times N \longrightarrow GL(n,\mathbb{C})$$
$$(k,a,n) \longmapsto kan.$$

$\boxed{\text{Example 2.19.}}$

It is common to restrict to matrices with det = 1.

We have. $SL(n,\mathbb{R}) := \{ g \in GL(n,\mathbb{R}) : \det g = 1 \}$

$SO(p,q) := O(p,q) \cap SL(n,\mathbb{R})$.

$SL(n,\mathbb{C}) := \{ g \in GL(n,\mathbb{C}) : \det g = 1 \}$

$SU(n) := U(n) \cap SL(n,\mathbb{C})$.

## 2.2 Compactness and local compactness.

We will be mostly focusing on locally compact groups. Key for us will be the existence of a Haar measure in the locally compact case..

**Definition 2.2.1**

A topological space $X$ is locally compact if every point admits a compact neigh.

**Lemma 2.21**

A Hausdorff space $X$ is locally compact iff every point admits a fundamental system of compact neighborhoods.

**Lemma 2.22**

Let $X$ be locally compact and Hausdorff. Then $Y \subset X$ is locally compact iff it is open in its closure.

**Example 2.23**

Any group $G$ with the discrete topology. $(\mathbb{R}^n, +)$, $\mathbb{R}^*$, $\mathbb{C}^*$ are locally compact and Hausdorff.

$GL(n, \mathbb{R})$ is loc. compact Hausdorff by Lemma 2.22

**Exercise 2.24.**

If $M$ is a manifold with $\dim M \geq 1$ then $\mathrm{Homeo}(M)$ is not locally compact

**Example 2.25**

Let $G_\alpha$, $\alpha \in A$ be a set of Hausdorff top. groups. Then

i). $\prod_{\alpha \in A} G_\alpha$ is compact iff $G_\alpha$ is compact $\forall \alpha \in A$.

ii). $\prod_{\alpha \in A} G_\alpha$ is loc compact iff,

all $G_\alpha$'s are loc. compact and all $G_\alpha$'s are compact except finitely many

Example 2.26

If $(X,d)$ is a proper metric space then $\mathrm{Iso}(X)$ is locally compact. If $(X,d)$ is compact then $\mathrm{Iso}(X)$ is compact.

This follows from

Theorem 2.27 (Ascoli-Arzelà)

$(X,d)$ metric space. $\mathcal{F} \subset C(X,X)$ has compact closure iff.

$\mathcal{F}$ is equicontinuous and for every $x \in X$ the set $\{f(x) : f \in \mathcal{F}\}$ has compact closure in $X$.

$\boxed{\text{Example } 2.28}$

$A, N, K$ are closed subgroups of $GL(n,\mathbb{R})$ hence locally compact,

We claim that $O(n)$ is compact.
$O(n)$ is closed because inverse image of $Id$ via the continuous map.

$$M_{n,n}(\mathbb{R}) \longrightarrow M_{n,n}(\mathbb{R})$$
$$A \longmapsto A^t A$$

Writing $X = (X_1, \dots, X_n)$ ⟶ columns of $X$.

the condition $X^t X = Id$ implies
$\|X_i\|^2 = 1$ for each $i = 1, \dots, n$.
hence $\sum_i \|X_i\|^2 = n$. Hence $O(n)$ is bounded. Hence compact.

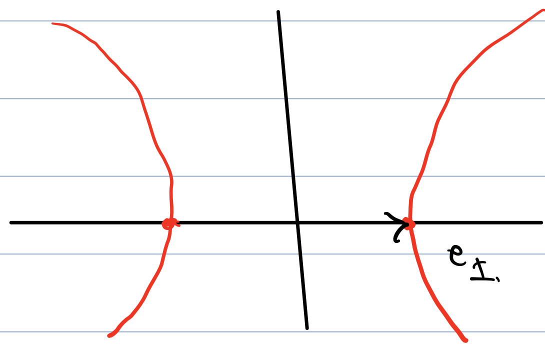
## Example 2.29

We mentioned that $O(p,q)$ is a closed subgroup of $GL(n,\mathbb{R})$. We claim that if $p \geq 1$ or $q \geq 1$ then $O(p,q)$ is non-compact.

To get an idea of why this is the case we consider $SO(1,1)$ which is a closed subgroup of $O(1,1)$

$$SO(1,1) = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} : \begin{array}{l} x^2 - y^2 = 1 \\ x,y \in \mathbb{R} \end{array} \right\}$$

We consider the orbit of $e_1 \in \mathbb{R}^2$ under the action of $SO(1,1)$ by multiplication



The hyperbola is clearly non-compact.
If $SO(1,1)$ was compact then the orbit $\left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} : x^2 - y^2 = 1, x, y \in \mathbb{R} \right\}$

should be compact. Contradiction.

$\boxed{\text{Exercise.}}$ Show that the connected component of the identity in $SO(1,1)$ is homeomorphic to $\mathbb{R}$.

$\boxed{\text{Exercise.}}$: Find a compact topological group with a non-compact subgroup.
$\rightsquigarrow$ Discuss solutions.

$\boxed{\text{Exercise}}$: Show that if $p, q \geq 1$ then $O(p,q)$ has a closed subgroup isomorphic to $\mathbb{R}$. Hence it is non-compact.

$\boxed{\text{Example 2.32}}$
Let $H$ be a complex and separable Hilbert space.

We endow $\mathcal{U}(H)$ (the group of unitary operators of $H$) with the strong operator topology. A basis of open sets is given by

$$\mathcal{U}(T; v_1, \dots, v_n; \varepsilon) :=$$
$$\{ S \in \mathcal{U}(H) : \| S v_i - T v_i \| < \varepsilon \quad 1 \le i \le n \}$$

It is possible to verify that $\mathcal{U}(H)$ is a topological group and

$\mathcal{U}(H)$ is locally compact iff $\dim H < \infty$

Moreover, if $\dim H < \infty$ then $\mathcal{U}(H)$ is compact.

## 2.3 General properties of topological groups

<u>Recall</u>: • a topological space is connected if it cannot be written as a disjoint union of two proper open subsets.

- the closure of a connected subset is connected.

- the continuous image of a connected set is connected.

Given a topological space $X$, we consider let $x \sim y$ if $\{x, y\} \subseteq$ connected subset of $X$.

The above defines an equivalence relation. The equivalence classes are called connected components of $X$.

The connected components are maximal connected subsets.

$\boxed{\text{Proposition 2.31.}}$

Let $G$ be a topological group. The following hold:

i) If $H \leq G$ is subgroup so is $\overline{H}$

ii) If $H \leq G$ is open then it is closed.

iii) The connected component $G^\circ$ of $G$ containing the neutral element is a closed normal subgroup of $G$

iv) If $G$ is connected and $U \ni e$ is a neighborhood of $e$ then $\bigcup_{n \geq 1} U^n = G$

v) If $G$ is connected and $N \trianglelefteq G$ is discrete and normal, then $N$ is contained in the center $Z(G)$

$\boxed{\text{Notation:}}$ Given subsets $A_1, \ldots, A_n \subseteq G$

$A_1 \cdots A_n = \{ a_1 \cdots a_n : a_i \in A_i \ \forall i \}$

Given $U \subseteq G$

$U^n := \{ u_1 \cdots u_n : u_i \in U \ \forall i \leq n \}$

$U^{-1} = \{ u^{-1} : u \in U \}$

We say that $V \subseteq G$ is symmetric if $V^{-1} = V$.

Before proving Prop 2.31. we state and prove a useful lemma.

**Lemma 2.32**

(i) If $U \ni e$ is a neigh. of $e$ then there exists $V \ni e$ symmetric and open with $V \subset U$

(ii). If $U \ni e$ is a neigh. of $e$ then there exists $V \ni e$ open symmetric with $V^2 = V \cdot V^{-1} \subset U$

**Proof**

(i). We can find $e \in W \subset U$ with $W$ open. $W^{-1} = i(W)$ is also open.
Then $V := W^{-1} \cap W \ni e$ is open, symmetric, with $V \subset U$.

(ii). Remember that $m : G \times G \to G$

is continuous by def. In particular it is continuous at $(e,e) \in G \times G$.
Hence there is $W \ni e$ neigh. of $e$.
s.t $W^2 = m(W \times W) \subset U$
It is sufficient to use (i) to find $V \ni e$ open symmetric with $V \subset W$

$\square$

## Proof of Prop. 2.31.

i). By continuity of $m$.

$$m(\overline{H \times H}) \subseteq \overline{m(H \times H)} = \overline{H}$$

$$\| \quad m(\overline{H} \times \overline{H}).$$

Since $i$ is a homeo. $G \circlearrowleft$

$$i(H) = H \implies i(\overline{H}) = \overline{H}$$

Hence $\overline{H}$ is a subgroup.

ii). We let $R$ be a set of representatives for $G/_H$ with $R \ni e$.

Then $G = H \sqcup \bigsqcup_{r \in R \setminus \{e\}} r \cdot H$.

Note that $rH = L_r(H)$ is open for each $r \in R$.

Hence $\bigsqcup_{r \in R \setminus \{e\}} r \cdot H$ is open.

Hence $H = G \setminus \bigsqcup_{r \in R \setminus \{e\}} rH$ is closed.

iii). $G^\circ \times G^\circ \ni (e, e)$ is connected.

Hence $m(G^\circ \times G^\circ) \ni e$ is connected.

$\Rightarrow m(G^\circ \times G^\circ) \subseteq G^\circ$

Moreover $i$ homeo with $i(e) = e$

$\Rightarrow i(G^\circ) = G^\circ$

$\Rightarrow G^\circ$ is a subgroup.

Since $\overline{G^\circ} \supset G^\circ \ni e$ is connected, by maximality $\overline{G^\circ} = G^\circ$, hence $G^\circ$ is closed.

For all $g \in G$ we get $\mathrm{int}(g) : G \longrightarrow G$
$x \longmapsto gxg^{-1}$

Note that $\text{int}(g)$ is clearly continuous.

$\text{int}(g)(e) = e. \Rightarrow \text{int}(g)(G^\circ) \subset G^\circ$

$\Rightarrow G^\circ$ is normal in $G$.

iv) Let $V = V^{-1} \ni e$. open with $V \subset U$

Then $H := \bigcup_{n \geq 1} V^n \subseteq \bigcup_{n \geq 1} U^n$

Claim: $H$ is a subgroup. (Exercise.)

Moreover each $V^n$ is open $\Rightarrow$ $H$ is open subgroup. Hence $H$ is closed by ii).

Hence $H = G \Rightarrow \bigcup_{n \geq 1} U^n = G$.

v). Let $N \trianglelefteq G$ be discrete normal.

$\forall n \in N$ we have a continuous map.

$$G \longrightarrow N$$
$$g \longmapsto gng^{-1}$$

composition of continuous maps.!

The image of this map is connected.

(because $G$ is connected), and it contains $\{n\}$. Since $N$ is discrete, the image is $\{n\}$. Hence $N \subset Z(G)$

$\square$

**Remark 2.33**

For a topological space $X$, $\pi_0(X)$ denotes the set of connected components.

By Prop. 2.31 (iii), when $G$ is a top group we can identify $\pi_0(G)$ with $G/G_0$.

Hence $\pi_0(G)$ acquires a group structure.

**Remark.** computing $\pi_0(G)$ can be (extremely) difficult, also for some innocent looking groups $G$.

**Example**

- If $T^2 = S^1 \times S^1$, then $\pi_0(\mathrm{Homeo}(T^2)) = GL(2, \mathbb{Z})$

More generally $\pi_0(\mathrm{Homeo}(T^n)) \simeq GL(n, \mathbb{Z})$

- $\pi_0(O(4,\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$ (Exercise)
- $O(4,\mathbb{R}) < \mathrm{Diff}(S^3)$

  $\pi_0(\mathrm{Diff}(S^3)) \simeq \mathbb{Z}/2\mathbb{Z}$ [Cerf, 1969]

We discuss now the relation of Prop 2.31 (v) with covering theory.

Fact: if $X$ is path connected, locally path-connected and semi locally simply connected then a universal covering of the pointed space $(X,*)$ always exists.

Exercise Let $H$ be a top. group and $p: G \longrightarrow H$ is a covering space. Assume that both $H$ and $G$ are locally path-connected and path connected. Then $\forall\, e^* \in p^{-1}(e)$ there exists a unique topological group structure on $G$

with neutral element $e^*$ such that $p: G \longrightarrow H$ is a homomorphism.

**Note:** • if we drop all the connectedness assumptions then there are counterexamples.

• if we only assume <u>connectedness</u> and not <u>path-connectedness</u> and <u>local path-connectedness</u> then the exercise seems to be an <u>open problem</u>.

**Definition:** a topological space $X$ is <u>locally path-connected</u> if for any neigh. $x \in U$ and every $x \in X$ there exists an open set $x \in V \subseteq U$ s.t. $V$ is path-connected.

**Definition:** a top. space $X$ is <u>semi locally simply connected</u> if for every $x \in X$ there is a neigh. $x \in U$ s.t. every loop in $U$ is null-homotopic in $X$.

If we assume that $H$ satisfies the assumptions of the Fact above, we can consider $p: \tilde{H} \longrightarrow H$ a universal covering map, after fixing $e^* \in p^{-1}(e)$ and the corresponding group structure on $\tilde{H}$.

Claim: $Ker(p) = p^{-1}(e) \subset Z(\tilde{H})$

Indeed: $Ker(p) = p^{-1}(e)$ is discrete and normal in the connected group $\tilde{H}$. Hence by Prop. 2.31 (v) it is contained in the center.

Fundamental group.

Since $Ker(p) \simeq \pi_1(H, e)$, we conclude that $\pi_1(H, e)$ is abelian.

## 2.4 Local homomorphisms.

This section will be fundamental later when we will discuss the correspondence between Lie algebras and Lie groups.

**Definition 2.34**

Let $G$, $H$ be top. groups. A local homo-
-morphism is a pair $(p, U)$ consisting
of a neighborhood $U$ of $e$ and a
continuous map $p : U \longrightarrow H$ s.t.
whenever $x, y, xy \in U$ then

$$p(xy) = p(x)\, p(y).$$

It is called a <u>local isomorphism</u> if
$p : U \longrightarrow p(U)$ is a <u>homeomorphism</u>.

**Example 2.35**:

It is easy to check that $(\mathbb{R}, +)$ and
$S^1 = T^1 = \{ z \in \mathbb{C} : |z| = 1 \} \subseteq \mathbb{C}^*$
are locally isomorphic.
They are not isomorphic. (clearly.)

**Example 2.36**

If $p : G \longrightarrow H$ is a covering homomorphism
of topological groups. then $G$ and $H$

are locally isomorphic. Exercise: explicit a local isomorphism from $H$ to $G$.

$\boxed{\text{Theorem 2.37}}$

If $\varphi : \underset{G}{\underset{\cap}{U}} \longrightarrow H$ is a local homomorphism and $G$ is path-connected and simply connected then $\varphi$ extends uniquely to a continuous homomorphism
$$G \longrightarrow H.$$

<u>Sketch of proof</u>

We discuss the strategy before giving some of the details. There will be three main steps:

<u>Step 1</u>: for any continuous path $\alpha : [0,1] \to G$ with $\alpha(0) = e$ we "extend $\varphi$ along $\alpha$" to define $\varphi(\alpha(1))$.

A priori it is not clear that the extension is well-defined in this way.

**Step 2**: we use that $G$ is simply connected to show that if $\alpha_0, \alpha_1 : [0,1] \longrightarrow G$ are continuous paths with $\alpha_0(0) = e = \alpha_1(0)$ and $\alpha_0(1) = \alpha_1(1)$ then $\rho(\alpha_0(1)) = \rho(\alpha_1(1))$.

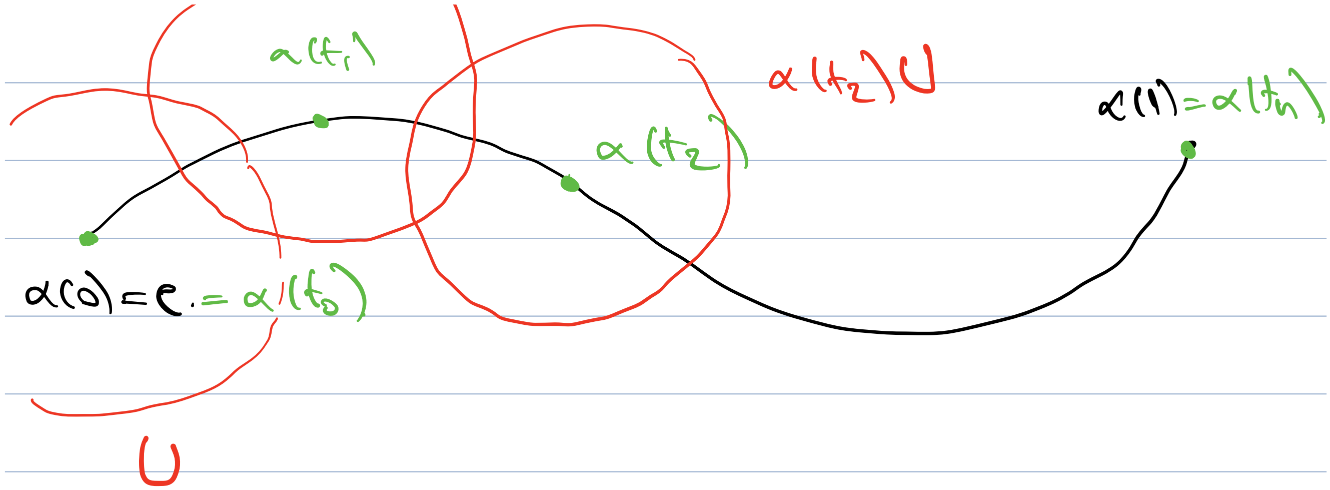
**Step 3**: we show that the extension $\rho : G \longrightarrow H$ obtained in this way is a continuous homomorphism.

Once we have done this, uniqueness follows by Prop. 2.31 (iv) (Exercise).

We discuss some details, starting from Step 1.

Let $\alpha : [0,1] \longrightarrow G$ be continuous with $\alpha(0) = e$. We say that a partition $t_0 = 0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ is good if $\forall\, s,t \in I_k = [t_{k-1}, t_k]$
$\alpha(s)^{-1}\alpha(t) \in U$

$\alpha(t_i)\, U$




**Claim 1**: good partitions exist.

Idea: use that $\alpha([0,1])$ is compact.

**Claim 2**: a refinement of a good partition is a good partition.

**Claim 3**: any two good partitions have a common refinement.

Given a good partition $0 = t_0 < \ldots < t_n = 1$, we observe that

$$\alpha(1) = \underset{\substack{\cap \\ U}}{\left(\alpha(t_0)^{-1}\alpha(t_1)\right)} \underset{\substack{\cap \\ U}}{\left(\alpha(t_1)^{-1}\alpha(t_2)\right)} \cdots \underset{\substack{\cap \\ U'}}{\left(\alpha(t_{n-1})^{-1}\alpha(t_n)\right)}$$

and define

$$\rho(\alpha(1)) := \rho(\alpha(t_0)^{-1}\alpha(t_1)) \cdots\cdots \rho(\alpha(t_{n-1})^{-1}\alpha(t_n))$$

Note that if we refine the partition by adding a single point $\bar{t} \in (t_{k-1}, t_k)$, then:

$$\underbrace{\alpha(t_{k-1})^{-1}\alpha(t_k)}_{\cap\, U} = \underbrace{\alpha(t_{k-1})^{-1}\alpha(\bar{t})}_{\cap\, U}\ \underbrace{\alpha(\bar{t})^{-1}\alpha(t_k)}_{\cap\, U},$$

hence, using that $\rho$ is a local homomorphism we get:

$$\rho(\alpha(t_{k-1})^{-1}\alpha(t_k)) = \rho(\alpha(t_{k-1})^{-1}\alpha(\bar{t}))\,\rho(\alpha(\bar{t})^{-1}\alpha(t_k)),$$

In particular, if we use this refinement to define $\rho(\alpha(1))$ we get the same result.

**Consequence:** $\rho(\alpha(1))$ is independent of the chosen partition of $[0,1]$ once we have fixed the path $\alpha$.

In **Step 2** we shall see that it is also independent of the choice of the path.

namely $\varphi_\alpha$ does not depend on $\alpha$.

Step 2: Let $\alpha_0, \alpha_1 : [0,1] \longrightarrow G$

with $\alpha_0(0) = \alpha_1(0) = e$, $\alpha_0(1) = \alpha_1(1) = g$.

$G$ is simply connected. Hence there exists a homotopy

$$H : [0,1] \times [0,1] \longrightarrow G$$

with $H(0,t) = \alpha_0(t)$, $H(1,t) = \alpha_1(t)$ $\forall\, t \in [0,1]$.

We can choose $W = W^{-1}$ neigh of $e$ s.t $W^2 \subset U$ by Lemma 2.32 (ii)

We set $\alpha_s : [0,1] \longrightarrow G$ by
$\alpha_s(t) := H(s,t)$, $\forall\, t \in [0,1]$
$\forall\, s \in [0,1]$. We also set $\varphi_s := \varphi_{\alpha_s}$.

We can find $\delta > 0$ s.t.

$$H(s_1,t_1)^{-1} H(s_2,t_2) \in W$$

for all $s_1, s_2, t_1, t_2$ with

$$|s_1 - s_2| + |t_1 - t_2| < \delta.$$

$(s_1,t_1,s_2,t_2) \longrightarrow H(s_1,t_1)^{-1} H(s_2,t_2)$ is a continuous map and $H(s,t)^{-1} H(s,t) = e$ $\forall\, s,t \in [0,1]$

Then for all $s \in [0,1]$ the partition $\{t_k\}_{k=0}^{n} = \{\frac{k}{n}\}_{k=0}^{n}$ is good if we choose $n$ large enough so that $\frac{1}{n} < \frac{\delta}{2}$.

We let $A := \{s \in [0,1] : p_s(g) = p_0(g)\}$.

Since $0 \in A \neq \emptyset$, it is enough to prove that A is open and closed.
It will follow that $A = [0,1]$ and hence $p_{\alpha_0}(\alpha_0(1)) = p_{\alpha_1}(\alpha_1(1))$.

End of lecture 3

We discuss how to prove that A is open. The proof that it is closed is similar and it is left as an Exercise.

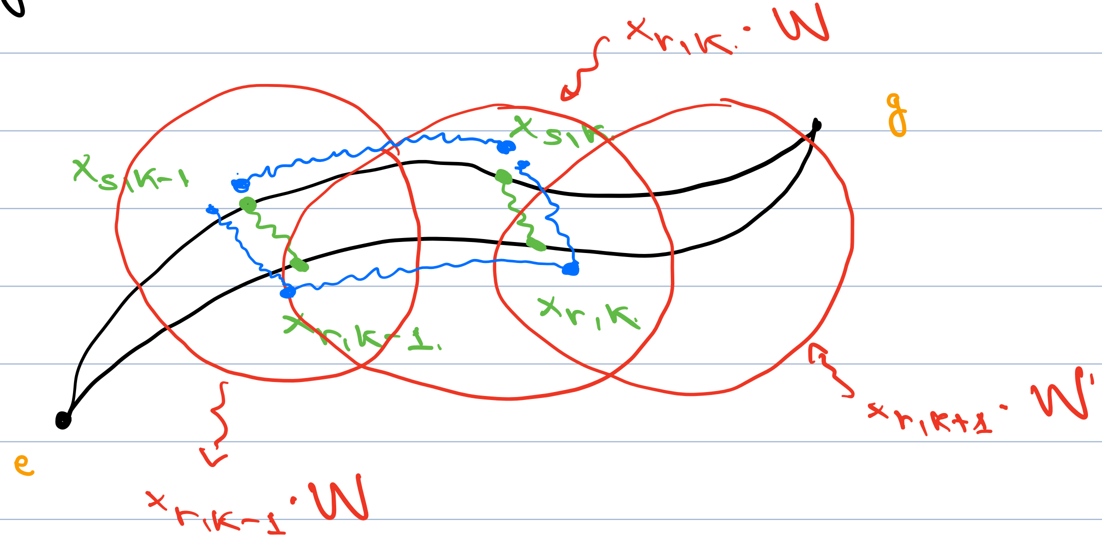
Denote $x_{s,k} := \alpha_s(t_k)$.

By construction:

$$\overline{x_{s,k-1}\, x_{s,k}} \in W' \qquad 1 \leq k \leq n,\ s \in [0,1].$$

We let $s, r$ be such that $|s-r| < \delta$.

and consider the images of the partition along the curves $\alpha_r$ and $\alpha_s$




Since $|s-r|<\delta$ we have.

$$y_k := x_{s,k}^{-1}\, x_{r,k} \in W \quad \forall k.$$

Then we can write.

$$\underbrace{x_{s,k-1}^{-1}\cdot x_{s,k}}_{\in\, W} = \underbrace{y_{k-1}}_{\in\, W}\cdot \underbrace{\underbrace{(x_{r,k-1}^{-1}\cdot x_{r,k})}_{\in\, W}\cdot \underbrace{y_k^{-1}}_{\in\, W'}}_{\in\, U}$$

using that $W^2 \subset U$.

By the <u>local homomorphism</u> property

$$\varphi(x_{s,k-1}^{-1} \cdot x_{s,k}) = \varphi(y_{k-1}) \cdot \varphi(x_{r,k-1}^{-1} \cdot x_{r,k}) \cdot \varphi(y_k)^{-1}$$

Then we compare the product defining $\varphi_{\alpha_s}(\alpha_s(1))$ with the one defining $\varphi_{\alpha_r}(\alpha_r(1))$.

$\varphi(y_0) = e.$

$$\prod_{k=1}^{n} \varphi(x_{s,k-1}^{-1} \cdot x_s) = \varphi(x_{r,0}^{-1} x_{r,1}) \cdot \varphi(y_1)^{-1} \varphi(y_1) \cdot$$

$$\cdot\, \varphi(x_{r,1}^{-1} x_{r,2}) \cdot \varphi(y_2) \cdots\cdots$$

$$\cdots \varphi(y_{n-1})^{-1} \varphi(y_{n-1}) \cdot \varphi(x_{r,n-1}^{-1} x_{r,n})\, \varphi(y_n)^{-1}$$

$$= \varphi(x_{r,0}^{-1} \cdot x_{r,1}) \cdot \varphi(x_{r,1}^{-1} \cdot x_{r,2}) \cdots \varphi(x_{r,n-1}^{-1} x_{r,n})$$

$$= \prod_{k=1}^{n} \varphi(x_{r,k-1}^{-1} \cdot x_{r,k})$$

since $\varphi(y_0) = \varphi(y_n) = e$,

This completes the proof that $A$ is open.

**Step 3**: In order to check that $\varphi$.

is a homomorphism we note that if $\alpha$ is a path from $e$ to $g$, and $\beta$ is a path from $e$ to $h$, then the concatenation of $\alpha$ with $g\beta$ is a path from $e$ to $gh$, and by construction of $\rho$ it holds

$$\rho(gh) = \rho(g)\cdot\rho(h).$$

See picture below.

The continuity then follows immediately from the continuity of $\rho$. $\square$

**Corollary 2.38**

Let $G$ be a path connected, locally path connected and semi locally simply connected top. group with $p : \tilde{G} \to G$ universal covering.

Let

$$\begin{array}{ccc} G & & \\ \cup\!\!\!\mid & & \\ U & \xrightarrow{\rho} & H \end{array}$$

be a local homomorphism, and $V \ni \tilde{e} \in \tilde{G}$ a neigh. of $\tilde{e}$ s.t. $p(V) \subseteq U$.

Then $\rho\circ p : V \to H$ extends to

a unique continuous homomorphism.

$$\tilde{G} \longrightarrow H.$$

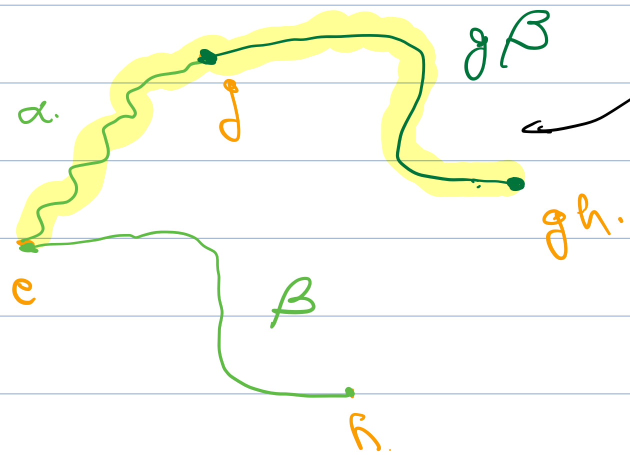
Note that this is a local homeomorphism.

$\boxed{\text{Note:}}$ $GL(n,\mathbb{Z}) = M_{n,n}(\mathbb{Z}) \cap \{\det = \pm 1\}$

$\boxed{\text{Note:}}$ $\pi_0(\mathrm{Diff}(S^4))$ is not known.

$\boxed{\text{Note.}}$ from now on, unless otherwise stated, if we say that $X$ is simply connected it will be implicit that $X$ is path connected.

<u>Picture for Step 3 of the proof above</u>




Concatenation of $\alpha$ and $g\beta$